Hamiltonian theory for the axial perturbations of a dynamical spherical background

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We develop the Hamiltonian theory of axial perturbations around a general time-dependent spherical background spacetime. Using the fact that the linearized constraints are gauge generators, we isolate the physical and unconstrained axial gravitational wave in a Hamiltonian pair of variables. Then, switching to a more geometrical description of the system, we construct the only scalar combination of them. We obtain the well-known Gerlach and Sengupta scalar for axial perturbations, with no known equivalent for polar perturbations. The strategy suggested and tested here will be applied to the polar case in a separate article.

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I. INTRODUCTION AND OVERVIEW

Perturbation Theory (both linear and higher order) is one of the most successful tools in General Relativity (GR). It has been used to find the stability properties of a large variety of background solutions like black holes, critical or cosmological solutions. It is also useful to model the evolution of dynamical processes in astrophysical scenarios that slightly deviate from an exact symmetry, like the oscillations of a static spherical neutron star or a nearly-spherical supernova explosion. In particular, it can be used to investigate the emission of gravitational waves in those processes.

A central problem in GR perturbation theory, inherited from the diffeomorphism invariance of the full theory, is that of isolating the physical degrees of freedom from the gauge-dependent information [1]. This can be done by imposing convenient *gauge fixing* conditions on the perturbations, as Regge and Wheeler [2] originally did in their study of perturbations of a Schwarzschild black hole. They and later Zerilli [3] succeeded in isolating the two physical degrees of freedom of the gravitational field around spherical vacuum, by taking suitable linear combinations of the remaining perturbations and their radial derivatives. These two variables further decouple due to their different properties under parity inversion: the Regge-Wheeler variable is axial and the Zerilli variable is polar.

A more systematic treatment of the gauge freedom in GR perturbation theory was pioneered by Moncrief [4] in his Hamiltonian study of the nonspherical perturbations of Schwarzschild. In a Hamiltonian context the four constraints obeyed by the twelve dynamical gravitational variables are the generators of the gauge transformations. Moncrief was able to use this information to perform several canonical transformations which reorganized the original six canonical pairs of variables into two physical pairs (equivalent to the Regge-Wheeler and Zerilli variables and their canonical momenta) and four gauge pairs in which the momenta were constrained to be zero, without any gauge fixing. The same technique was later applied to other spherical backgrounds with additional symmetries, like Reissner-Nordström [5, 6], Oppenheimer-Snyder [7] or Friedmann-Robertson-Walker [8], but has never been applied to general spherically symmetric backgrounds, possibly highly time dependent. Hamiltonian perturbation theory has also been recently revisited in Quantum Gravity with a cosmological background [9]. A drawback of the Hamiltonian approach is that it is tied to a particular foliation of the background spacetime, and hence the geometric properties of the gauge-invariant variables under coordinate transformations involving time are far from obvious.

A *Lagrangian* formalism was introduced by Gerlach and Sengupta [10] (GS) to study perturbations around generic spherical spacetimes, in which the metric perturbation is geometrically split along the decomposition of the 4d manifold M^4 into the product of a general 2d Lorentzian manifold M^2 with boundary and the unit 2-sphere S^2. This is a highly geometrical framework, in which the meaning of the perturbations is transparent, and which also allows the construction of gauge-invariant variables. In the axial case it has been possible to isolate the gravitational degree of freedom in a single scalar *master* variable which obeys a wave equation and can be coupled to any kind of matter, both in the background and the perturbations. This master scalar and its equation generalize the Regge-Wheeler variable and its equation to the axial perturbative problem around spherical symmetry for any reasonable matter model, and hence can be considered as the optimal framework for a perturbative study. Unfortunately, in the polar case there is not a master scalar valid for a generic spherical background and any matter model, though there are results for some particular cases. For instance, a master Zerilli scalar has been introduced by Sarbach and Tiglio [11] for a Schwarzschild background, which was later generalized to nonlinear electrodynamics [12], around any background

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solution of the theory. In references [13, 14] the gauge-invariant combinations of the stress-energy tensor were also included but still on a vacuum background.

Both approaches to metric perturbation theory are complementary: the Hamiltonian approach offers a better framework to handle gauge-invariance, while the Lagrangian approach gives a clearer picture of the geometrical structures being perturbed. This Article proposes a combination of both formalisms to construct gauge-invariant scalar perturbative variables containing all physically relevant information concerning the gravitational waves. We restrict ourselves to spherical backgrounds, but which can be highly dynamical. For definiteness, the dynamics will be introduced using a real massless scalar field, but could be done through any other matter model admitting a Hamiltonian description. This Article focuses on the axial subset of perturbations, for which the sought solution is the Gerlach and Sengupta master scalar [10], previously found using the Lagrangian method only. We show how the Hamiltonian way allows a more systematic derivation of this object, and how both approaches mutually relate. Most important, this Article prepares the path towards a systematic analysis of the polar problem, for which a general gauge-invariant master scalar has never been found. Such analysis will be discussed in a second publication.

The Article is organized as follows. Section II presents a brief review of the ADM formalism, establishing the notations for 3d and 4d objects, both in the background and first-order perturbations. Section III particularizes to a spherical background. Section IV restricts to axial perturbations and carries out the complete program of scalar gauge-invariant construction, as well as establishing the connection between the Hamiltonian and Lagrangian approaches to the problem. We conclude in Section V with some remarks.

II. HAMILTONIAN PERTURBATIONS IN GENERAL RELATIVITY

A. ADM Hamiltonian formalism

Given the four-dimensional spacetime $(M^4, {}^{(4)}g_{\mu\nu})$, we introduce a foliation of 3d spacelike slices as level surfaces of the time field $t(x)$. The orthogonal vector u^μ defines the projected metric ${}^{(3)}g_{\mu\nu} = {}^{(4)}g_{\mu\nu} + u_\mu u_\nu$ on the slices. We introduce coordinates (t, x^i) adapted to the foliation, and work with three-dimensional objects. Only in the last part of this Article we shall use four-dimensional metric variables in order to compare our results with those from the Gerlach and Sengupta formalism. Greek and Latin indices denote 4d and 3d tensors respectively. A left-superindex indicates dimensionality when confusion may arise.

The 4-metric is decomposed as customary in the lapse function, the shift vector and the 3-metric on the slices

$$\alpha^{-2} \equiv -{}^{(4)}g^{tt}, \qquad \beta_i \equiv {}^{(4)}g_{ti}, \qquad g_{ij} \equiv {}^{(4)}g_{ij}, \tag{1}$$

with inverse

$$g^{ij} = {}^{(4)}g^{ij} + \alpha^{-2}\beta^i\beta^j, \tag{2}$$

with Latin indices always raised and lowered with g^{ij} and g_{ij}.

The gravitational dynamical variables in the ADM Hamiltonian formalism are g_{ij} and their conjugated momenta:

$$\Pi^{ij} \equiv \mu_g \left(g^{ij} K^l{}_l - K^{ij} \right), \qquad \mu_g \equiv \sqrt{\det g_{ij}}, \tag{3}$$

where K^{ij} is the extrinsic curvature of the foliation hypersurfaces.

The spacetime will be assumed to contain a dynamical Klein-Gordon field Φ, whose evolution is controlled by the action

$$\mathcal{S}_{KG} = -\frac{1}{2}\int dx^4 \sqrt{-{}^{(4)}g}\ {}^{(4)}g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu} \tag{4}$$

$$= \int dx^4 \left[\Pi\Phi_{,t} - \frac{\alpha}{2}\left(\frac{\Pi^2}{\mu_g} + \mu_g g^{ij}\Phi_{,i}\Phi_{,j} \right) - \beta^i \left(\Pi\Phi_{,i}\right) \right]. \tag{5}$$

Its canonical momentum has been defined as

$$\Pi \equiv -\sqrt{-{}^{(4)}g}\ {}^{(4)}g^{t\mu}\Phi_{,\mu}, \tag{6}$$

and ${}^{(4)}g$ denotes the determinant of the 4-metric. The complete action of the system, with coupling constant $16\pi G_N = 1$ following [4], is given by

$$\mathcal{S} = \mathcal{S}_G + \mathcal{S}_{KG} = \int dt \int d^3x \left(\Pi^{ij} g_{ij,t} + \Pi\Phi_{,t} - \alpha\mathcal{H} - \beta^i\mathcal{H}_i \right). \tag{7}$$

The Lagrange multipliers α and β^i are associated to the constraints

$$\mathcal{H} = \frac{1}{\mu_g}\left[\Pi^{ij}\Pi_{ij} - \frac{1}{2}\left(\Pi^l{}_l\right)^2\right] - \mu_g{}^{(3)}R + \frac{1}{2}\left(\frac{\Pi^2}{\mu_g} + \mu_g g^{ij}\Phi_{,i}\Phi_{,j}\right), \tag{8}$$

$$\mathcal{H}_i = -2D_j\Pi_i{}^j + \Pi\Phi_{,i}, \tag{9}$$

where D_j is the covariant derivative associated to g_{ij}. Variation of the action (7) with respect to g_{ij}, Π_{ij}, Φ and Π gives the evolution equations for the corresponding conjugated variables.

B. Hamiltonian metric perturbations

Now suppose that the whole system is perturbed at first order. We define the following special notations:

$$C \equiv \delta\alpha, \qquad B^i \equiv \delta(\beta^i), \tag{10}$$

$$h_{ij} \equiv \delta(g_{ij}), \qquad p^{ij} \equiv \delta(\Pi^{ij}), \tag{11}$$

$$\varphi \equiv \delta\Phi, \qquad p \equiv \delta\Pi. \tag{12}$$

Notice that we perturb the contravariant components of the shift vector because this will give rise to simpler equations, even though the comparison with GS variables will be slightly more involved because β_i is better related to the 4-metric [see Eq. (1)].

Following Taub [15] and Moncrief [4] we shall obtain the equations for the linear perturbations using the Jacobi method of second variations. The idea is that the second variation of the action (7), keeping only terms that are quadratic on first-order perturbations, gives an action functional for the perturbations,

$$\frac{1}{2}\delta^2\mathcal{S} = \int dx^4\left[p^{ij}h_{ij,t} + p\varphi_{,t} - C\delta(\mathcal{H}) - B^i\delta(\mathcal{H}_i) - \frac{\alpha}{2}\delta^2(\mathcal{H}) - \frac{\beta^i}{2}\delta^2(\mathcal{H}_i)\right]. \tag{13}$$

There are three kinds of terms. First we have kinetic terms, containing time derivatives of h_{ij} and φ. Then we have the first variations of the constraints that, under a variation of the effective action (13) with respect to B^i and C, give the constraints that must be obeyed by the perturbations,

$$\delta(\mathcal{H}) = 0, \qquad \delta(\mathcal{H}_i) = 0. \tag{14}$$

And finally we have the second variations of the constraints, which are quadratic in the perturbations $(h_{ij}, p^{ij}, \varphi, p)$, and will give the evolution equations for those perturbations. Even though we started with an exact Hamiltonian which was a linear combination of constraints, we end up having a quadratic Hamiltonian which does not vanish on shell.

The constraints $\mathcal{H}$ and $\mathcal{H}_i$ in General Relativity are first-class constraints, and hence generators of gauge transformations on the constraint surface in phase space. This identifies the gauge orbits, but in general it is not possible to separate explicitly the 2 physical degrees of freedom (4 functions) from the 4 gauge variables and the 4 constrained variables in g_{ij} and Π^{ij}. The situation in the linearized theory is simpler, but still only highly symmetric background scenarios allow the construction of *gauge-invariant* algebraic combinations of perturbations and their derivatives containing the physical information in the linearized approximation. (See [1] for a discussion of the importance of symmetry on the algebraic character of the combinations.) One of such background scenarios is a spherically symmetric spacetime, as we shall exploit for the rest of this Article.

For completeness, we provide the expressions for the first variations of the constraints:

$$\begin{aligned}\delta(\mathcal{H}) &= \frac{1}{\mu_g}\left(\Pi_{ij} - \frac{1}{2}g_{ij}\Pi^l{}_l\right)\left(2p^{ij} + 2h^i{}_k\Pi^{kj} - \frac{1}{2}h^k{}_k\Pi^{ij}\right) \\ &+ \mu_g\left({}^{(3)}G^{ij}h_{ij} - D^iD^jh_{ij} + D^jD_jh^i{}_i\right) \\ &+ \frac{1}{4}h^k{}_k\left(-\frac{\Pi^2}{\mu_g} + \mu_g\Phi_{,i}\Phi^{,i}\right) + \frac{p\Pi}{\mu_g} + \mu_g\varphi_{,i}\Phi^{,i} - \frac{1}{2}\mu_g h^{ij}\Phi_{,i}\Phi_{,j},\end{aligned} \tag{15}$$

$$\delta(\mathcal{H}_i) = -2D_k(h_{ij}\Pi^{jk} + g_{ij}p^{jk}) + \Pi^{jl}D_ih_{jl} + p\Phi_{,i} + \Pi\varphi_{,i}. \tag{16}$$

See [16] for intermediate expressions and techniques to compute these expressions. The corresponding expressions for their second variations are

$$\delta^2(\mathcal{H}) = \frac{1}{8\mu_g}\{8p^2 + 8\mu_g^{2\,(3)}G^{ij}(h_{ij}h_k{}^k - 2h_i{}^kh_{jk}) + 16P_{ij}P^{ij} - 8P_i{}^iP_j{}^j \tag{17}$$

$$
\begin{aligned}
&+\ 2h^{ij}\left[h^{kl}(8\Pi_{ik}\Pi_{jl}-4\Pi_{ij}\Pi_{kl})+h_{ij}\left(\Pi^2-2\mu_g^{2\,(3)}R+2\Pi_{kl}\Pi^{kl}-\Pi_k{}^k\Pi_l{}^l\right)\right.\\
&-\ 8\mu_g^2(D_jD_ih_k{}^k-D_jD_kh_i{}^k-D_kD_jh_i{}^k+D_kD^kh_{ij})-8\Pi_k{}^kP_{ij}+32\Pi_i{}^kP_{jk}-8\Pi_{ij}P_k{}^k\Big]\\
&+\ h_i{}^i\left[h_j{}^j\left(\Pi^2+2\Pi_{kl}\Pi^{kl}-\Pi_k{}^k\Pi_l{}^l+2\mu_g^{2\,(3)}R\right)+8h^{jk}\left(\Pi_{jk}\Pi_l{}^l-2\Pi_j{}^l\Pi_{kl}\right)\right.\\
&-\ 8\mu_g^2(D_kD_jh^{jk}-D_kD^kh_j{}^j)-8\Pi p-16\Pi^{jk}P_{jk}+8\Pi_j{}^jP_k{}^k\Big]\\
&+\ \mu_g^2[8D_i\varphi D^i\varphi+h_j{}^jD_i\Phi(8D^i\varphi+h_k{}^kD^i\Phi)+2h_{jk}D^i\Phi(4h_i{}^kD^j\Phi-h^{jk}D_i\Phi)\\
&+\ 4D_jh_k{}^kD^jh_i{}^i-4h_{ij}(4D^i\varphi+h_k{}^kD^i\Phi)D^j\Phi+16(D_ih^{ij}-D^jh_i{}^i)D_kh_j{}^k+4(2D_jh_{ik}-3D_kh_{ij})D^kh^{ij}]\},
\end{aligned}
$$

$$
\delta^2(\mathcal{H}_i) = 2p\varphi_{,i}+2p^{jk}(D_ih_{jk}-2D_kh_{ij})-4h_{ij}D_kp^{jk}. \tag{18}
$$

III. SPHERICAL BACKGROUND

Let us know restrict to a spherically symmetry background $M^4 = M^2 \times S^2$, where S^2 is the unit two-sphere and M^2 is a two-dimensional Lorentzian manifold with boundary. We shall use arbitrary coordinates $x^A = (t, \rho)$ on M^2 and the usual spherical coordinates $x^a = (\theta, \phi)$ on S^2. Uppercase Latin indices $A, B, C, ...$ denote objects on M^2 and lowercase Latin indices $a, b, c, ...$ denote objects on S^2. The fact that we use arbitrary coordinates on M^2 will later allow us to keep track of the tensorial character of the different variables. Therefore we do not impose any condition on the lapse or shift, apart from being consistent with spherical symmetry.

The 4-metric can be 2+2 decomposed as

$$
(ds^2)_4 = g_{AB}(x^D)\,dx^Adx^B + r^2(x^D)\,d\Omega^2, \tag{19}
$$

with $d\Omega^2$ the round metric of the 2-sphere and g_{AB} and r being a metric field and a scalar field on M^2, respectively. We define the vector field $v_A \equiv r^{-1}r_{,A}$.

Using the radial coordinate ρ explicitly we can write the background spatial 3-metric as

$$
(ds^2)_3 = a^2(t,\rho)d\rho^2 + r^2(t,\rho)d\Omega^2. \tag{20}
$$

With a spherically symmetric lapse $\alpha = \alpha(t, \rho)$ and shift vector $\beta^i = (\beta(t, \rho), 0, 0)$ we have

$$
\begin{aligned}
(ds^2)_4 &= (-\alpha^2+a^2\beta^2)dt^2+2a^2\beta dtd\rho+(ds^2)_3 &(21)\\
&= -\alpha^2dt^2+a^2(d\rho+\beta dt)^2+r^2d\Omega^2, &(22)
\end{aligned}
$$

which takes the following matricial form,

$$
g_{AB} = \begin{pmatrix} -\alpha^2+a^2\beta^2 & a^2\beta \\ a^2\beta & a^2 \end{pmatrix}, \qquad g^{AB} = \begin{pmatrix} -\alpha^{-2} & \alpha^{-2}\beta \\ \alpha^{-2}\beta & a^{-2}-\alpha^{-2}\beta^2 \end{pmatrix}. \tag{23}
$$

The normal vector to the surfaces of constant t is $u_\mu = (-\alpha, 0, 0, 0)$ or $u^\mu = \alpha^{-1}(1, -\beta, 0, 0)$. Its orthogonal, radial vector is $n^\mu = (0, a^{-1}, 0, 0)$ or $n_\mu = a(\beta, 1, 0, 0)$. In order to work with more geometrical objects we define the following frame derivatives that act on any scalar field f:

$$
f' = n^\mu f_{,\mu} = \frac{f_{,\rho}}{a}, \qquad \dot{f} = u^\mu f_{,\mu} = \frac{f_{,t}-\beta f_{,\rho}}{\alpha}. \tag{24}
$$

We now derive the background equations, that will be later used to simplify the coefficients of the equations for the perturbations. It is convenient to define the following momentum-like variables, which have a definite tensorial character with respect to changes of the ρ coordinate,

$$
\Pi_1 \equiv \frac{a^2\Pi^{\rho\rho}}{\mu_g}, \qquad \Pi_2 \equiv \frac{2r^2\Pi^{\theta\theta}}{\mu_g}, \qquad \Pi_3 \equiv \frac{\Pi}{\mu_g}. \tag{25}
$$

We can write the constraints in terms of these spherical variables,

$$
\begin{aligned}
\frac{\mathcal{H}}{\mu_g} &= \Pi_1\left(\frac{\Pi_1}{2}-\Pi_2\right)-{}^{(3)}R+\frac{1}{2}\left({\Pi_3}^2+\Phi'^2\right)=0, &(26)\\
\frac{1}{a}\frac{\mathcal{H}_\rho}{\mu_g} &= -\frac{2}{r^2}(r^2\Pi_1)'+\frac{2r'}{r}\Pi_2+\Pi_3\Phi'=0, &(27)
\end{aligned}
$$

so that the action is

$$\frac{1}{4\pi}\mathcal{S} = \int dt \int d\rho\, ar^2 \left[2\Pi_1 \frac{a_{,t}}{a} + 2\Pi_2 \frac{r_{,t}}{r} + \Pi_3 \Phi_{,t} - \alpha \frac{\mathcal{H}}{\mu_g} - \beta \frac{\mathcal{H}_\rho}{\mu_g}\right]. \tag{28}$$

The evolution equations can be obtained by simple variation with respect to different variables:

$$\frac{1}{\alpha}\left[a_{,t} - (\beta a)_{,\rho}\right] = \frac{a}{2}\left(\Pi_1 - \Pi_2\right), \tag{29}$$

$$\frac{1}{\alpha}(r_{,t} - \beta r_{,\rho}) = -\frac{r}{2}\Pi_1, \tag{30}$$

$$\frac{1}{\alpha}(\Phi_{,t} - \beta \Phi_{,\rho}) = \Pi_3 \tag{31}$$

$$\frac{1}{\alpha}\left(\Pi_{1,t} - \beta \Pi_{1,\rho}\right) = \frac{3\Pi_1^2}{4} + \frac{1}{r^2} - \frac{r'}{r}\frac{(\alpha^2 r)'}{\alpha^2 r} + \frac{1}{4}\left(\Pi_3^2 + \Phi'^2\right), \tag{32}$$

$$\frac{1}{\alpha}\left(\Pi_{2,t} - \beta \Pi_{2,\rho}\right) = \frac{1}{2}(\Pi_1^2 + \Pi_2^2 - \Pi_1\Pi_2) + \frac{2\alpha' r'}{\alpha r} - \frac{2(\alpha r)''}{\alpha r} + \frac{1}{2}\left(\Pi_3^2 - \Phi'^2\right), \tag{33}$$

$$\frac{1}{\alpha}(\Pi_{3,t} - \beta \Pi_{3,\rho}) = \frac{\Pi_3(\Pi_1 + \Pi_2)}{2} + \frac{(\alpha r^2 \Phi')'}{\alpha r^2}. \tag{34}$$

Note that in spherical symmetry the restriction to vacuum, choosing Schwarzschild coordinates (t, $r = \rho$), is given by $\Pi^{\mu\nu} = 0$, ${}^{(3)}R = 0$ and $\Phi = 0$, that is, $\Pi_1 = \Pi_2 = \Pi_3 = \Phi = 0$, which simplifies the previous expressions. In particular, the constraints (26) and (27) are then trivially obeyed. This is the case studied by Moncrief [4], but here we want to analyze the general case.

IV. AXIAL PERTURBATIONS

A. Harmonic expansions

The tensor spherical harmonics form a complete basis on the 2-sphere to expand a tensor field of any rank. Appendix A gives the definitions for the harmonics we shall need in this Article, as well as some of their basic properties and the relations with the harmonics used by Moncrief [4]. See Ref. [16] for full definitions in the arbitrary rank case and further properties. Tensor harmonics can be separated in two groups according to their polarity: there are *polar* (or *even*, *electric* or *poloidal*) harmonics and *axial* (or *odd*, *magnetic* or *toroidal*) harmonics. In first-order perturbation theory around a spherical spacetime polar and axial harmonics decouple and this Article will only deal with the axial part of the problem. Following Regge-Wheeler's notation [2] for the metric perturbations and Moncrief's notations [4] for the momentum and shift vector, we expand the perturbative variables in tensor spherical harmonics:

$$h_{ij}dx^i dx^j = \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left\{-2(h_1)_l^m\, d\rho\, X_l^m{}_a\, dx^a + (h_2)_l^m\, X_l^m{}_{ab}\, dx^a dx^b\right\}, \tag{35}$$

$$\frac{1}{\mu_g}p_{ij}dx^i dx^j = \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left\{-2(\hat{p}_1)_l^m\, d\rho\, X_l^m{}_a\, dx^a + (\hat{p}_2)_l^m\, X_l^m{}_{ab}\, dx^a dx^b\right\}, \tag{36}$$

$$B_i dx^i = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} -(h_0)_l^m\, X_l^m{}_a\, dx^a, \tag{37}$$

$$C = 0, \tag{38}$$

$$p = 0, \tag{39}$$

$$\varphi = 0. \tag{40}$$

Note that there are no axial perturbations of the 3d scalars α, Φ and Π. That means that the scalar field plays no role from the perturbative point of view, though the background scalar field is still instrumental to allow for a general dynamical spacetime. As we will see, this does not imply any loss of generality. Different (l, m) harmonic components also decouple around spherical symmetry, and so from now on we shall drop them from the perturbative variables, assuming that we work with a fixed pair of labels at any time. It is important to note that h_2, $\hat{p}_2$ and h_0 are scalars under changes of the ρ coordinate, but h_1 and $\hat{p}_1$ behave as components of a vector. In the language of 1d spacetimes,

the latter are densities of weight +1, to be compensated with metric factors a to convert them into scalars. This will become clearer when comparing with the more geometrical GS approach.

The variables (h_1, p_1) and (h_2, p_2) form two pairs of canonically conjugated variables, whose evolution is partially determined by the arbitrary function h_0. For example the evolution equations for the variables h_1 and h_2 can be easily obtained by perturbation of formula (3) after introducing the expansions (35–40)

$$\frac{1}{\alpha}\left(h_{1,t} - (\beta h_1)_{,\rho}\right) = 2\hat{p}_1 + \Pi_1 h_1 + \frac{r^2}{\alpha}\left(\frac{h_0}{r^2}\right)_{,\rho}, \tag{41}$$

$$\frac{1}{\alpha}\left(h_{2,t} - \beta h_{2,\rho}\right) = 2\hat{p}_2 + (\Pi_2 - \Pi_1) h_2 - \frac{2h_0}{\alpha}. \tag{42}$$

We shall later obtain the evolution equations for more convenient momenta variables.

B. Gauge-invariant perturbations

The action functional for the axial perturbations is

$$\frac{1}{2}\left(\delta^2 \mathcal{S}\right)^{\text{axial}} = \int dt \int dx^3 \left[p^{ij} h_{ij,t} - B^i \delta(\mathcal{H}_i) + ...\right]^{\text{axial}} \tag{43}$$

$$= \int dt \left\{\int d\rho \ \left(p_1 h_{1,t} + p_2 h_{2,t}\right) + H[h_0] + ...\right\}, \tag{44}$$

where the dots denote those terms coming from the second variation of the constraints, which we do not need to consider in this subsection. The functional H will be defined below in terms of the first variation of the constraint. We have also defined

$$p_1 = \frac{2l(l+1)}{a}\hat{p}_1^*, \qquad p_2 = \frac{a\lambda}{r^2}\hat{p}_2^*, \tag{45}$$

where the star stands for complex conjugation and we have defined

$$\lambda \equiv \frac{1}{2}(l-1)l(l+1)(l+2). \tag{46}$$

In term of these variables, the perturbed constraint is given by

$$\delta(\mathcal{H}_a)^{\text{axial}} = \frac{X_a \mu_g}{l(l+1)}\left\{\frac{(r^2 p_1)_{,\rho}}{ar^2} + 2\frac{p_2}{a} + \lambda\frac{\Pi_2 h_2}{r^2} + \frac{2l(l+1)}{ar^2}\left(\frac{r^2 \Pi_1 h_1}{a}\right)_{,\rho}\right\} \tag{47}$$

which in turn defines the functional

$$H[h_0] = -\int dx^3 B^i \delta(\mathcal{H}_i)^{\text{axial}} \tag{48}$$

$$= \int d\rho \left\{-r^2\left(\frac{h_0}{r^2}\right)_{,\rho} p_1 + 2h_0 p_2 + \lambda a \Pi_2 \frac{h_0}{r^2} h_2 - \frac{2l(l+1)}{a} r^2 \Pi_1 \left(\frac{h_0}{r^2}\right)_{,\rho} h_1\right\}. \tag{49}$$

This functional is the generator of gauge transformations, and of course commutes with itself on shell,

$$[\,H[f], H[g]\,] = \frac{1}{l(l+1)}\int d\rho \left\{r^4 (f_{,\rho} g - g_{,\rho} f)\frac{1}{a}\frac{\mathcal{H}_\rho}{\mu_g}\right\}, \tag{50}$$

for arbitrary scalar fields f and g.

Following Moncrief [4] we perform two canonical transformations to separate the gauge-invariant information from the pure-gauge content in the canonical pairs (h_1, p_1) and (h_2, p_2). The first canonical transformation constructs the gauge-invariant combination k_1, also a vector component,

$$k_1 \equiv h_1 + \frac{r^2}{2}\left(\frac{h_2}{r^2}\right)_{,\rho}, \qquad k_2 \equiv h_2. \tag{51}$$

It induces the following transformation on the momenta:

$$\pi_1 = p_1, \qquad \pi_2 = p_2 + \frac{(r^2 p_1)_{,\rho}}{2r^2}, \tag{52}$$

and can be obtained from the generating function

$$F(p_1, p_2, k_1, k_2) = p_1 k_1 + p_2 k_2 - p_1 \frac{r^2}{2} \left(\frac{k_2}{r^2}\right)_{,\rho}. \tag{53}$$

In terms of the new variables we can write the first variation of the axial constraint as:

$$\delta(H_a)^{\rm axial} = \frac{X_a \mu_g}{l(l+1)} \left\{ 2\frac{\pi_2}{a} + \lambda \frac{\Pi_2 k_2}{r^2} + \frac{2l(l+1)}{ar^2} \left[\frac{r^2 \Pi_1}{a} \left(k_1 - \frac{r^2}{2} \left(\frac{k_2}{r^2}\right)_{,\rho} \right) \right]_{,\rho} \right\}, \tag{54}$$

which does not contain π_1 and therefore commutes with k_1. That is, k_1 is gauge-invariant, as we had anticipated. This suggests the second canonical transformation:

$$Q_1 \equiv k_1, \qquad Q_2 \equiv k_2, \tag{55}$$

with conjugated momenta

$$P_1 \equiv \pi_1 - l(l+1) \frac{r^2 \Pi_1}{a} \left(\frac{k_2}{r^2}\right)_{,\rho}, \tag{56}$$

$$P_2 \equiv \pi_2 + \frac{\lambda}{2r^2} a \Pi_2 k_2 + \frac{l(l+1)}{r^2} \left[\frac{r^2 \Pi_1}{a} \left(k_1 - \frac{r^2}{2} \left(\frac{k_2}{r^2}\right)_{,\rho} \right) \right]_{,\rho}. \tag{57}$$

We can obtain this canonical transformation from the generating function

$$F(P_1, P_2, k_1, k_2) = P_1 k_1 + P_2 k_2 + a\, l(l+1) \left\{ \frac{r^2 \Pi_1}{a} \left(\frac{k_2}{r^2}\right)_{,\rho} \frac{k_1}{a} - \Pi_1 \left[\frac{r^2}{2a} \left(\frac{k_2}{r^2}\right)_{,\rho} \right]^2 - \frac{(l-1)(l+2)}{8r^2} \Pi_2 k_2^2 \right\}. \tag{58}$$

The first canonical transformation is independent of the dynamical content of the background spacetime, in the sense that it does not contain the background momenta Π_1, Π_2, Π_3. It is actually identical to that of Moncrief [4]. For the sake of clarity, we have separated the influence of the dynamical background into the second canonical transformation, which trivializes for any static background.

At this point we have isolated the physical information of the axial metric perturbation in the pair (Q_1, P_1) while the (Q_2, P_2) contains the gauge subsystem. P_2 is the generator of gauge transformations,

$$\delta(\mathcal{H}_a)^{\rm axial} = \frac{X_a \mu_g}{l(l+1)} \frac{2P_2}{a} \tag{59}$$

and hence it is gauge-invariant but constrained to vanish. Its conjugated variable Q_2 is gauge-dependent and its time evolution is determined by the arbitrary function h_0, which can be used to set any desired value for $Q_{2,t}$.

C. Evolution equations

After replacing the new variables and integrating by parts a number of times, we get the following Jacobi action:

$$\frac{1}{2} \left(\delta^2 \mathcal{S}\right)^{\rm axial} = \int dt \int d\rho \, \left[P_1 (Q_{1,t} - (\beta Q_1)_{,\rho}) + P_2 (Q_{2,t} - \beta Q_{2,\rho}) + 2 P_2 h_0 - \alpha \mathcal{H}^{(1)} \right], \tag{60}$$

where we have defined the first-order quadratic Hamiltonian

$$\begin{aligned} \mathcal{H}^{(1)} \equiv\ & \Pi_1 (P_1 Q_1 - P_2 Q_2) + \frac{1}{a\lambda r^2} \left[\left(\frac{r^2 P_1}{2} + l(l+1) \frac{r^2 \Pi_1}{a} Q_1 \right)_{,\rho} - r^2 P_2 \right]^2 \\ & + \frac{a P_1^2}{2l(l+1)} + \frac{l(l+1)}{a} \left[\frac{(l-1)(l+2)}{2r^2} + \frac{\Pi_1 (\Pi_2 - \Pi_1)}{2} + \dot{\Pi}_1 \right] Q_1^2. \end{aligned} \tag{61}$$

The variation of the action with respect to h_0 gives the constraint that must be obeyed by the perturbations. This constraint now takes the simple form

$$P_2 = 0. \tag{62}$$

This constraint is conserved in the evolution since variation with respect to Q_2 gives

$$(r^2 P_2)_{,t} = (\beta r^2 P_2)_{,\rho}. \tag{63}$$

As P_2 is the generator of the gauge transformations, its conjugated variable Q_2 is pure gauge. Its evolution equation comes from taking the variation of the action respect to P_2,

$$\frac{1}{\alpha}(Q_{2,t} - \beta Q_{2,\rho}) = -2\frac{h_0}{\alpha} - \Pi_1 Q_2 - \frac{1}{a\lambda}\left(r^2 P_1 + l(l+1)\frac{2r^2\Pi_1}{a}Q_1\right)_{,\rho}. \tag{64}$$

The initial data for Q_2 is gauge, and its evolution is fully determined by the free function h_0. In particular it is possible to choose $Q_2 = 0$ initially and take h_0 so that $Q_2 = 0$ at all times.

We can obtain the physically relevant equations by variation of the action with respect to the variables (Q_1, P_1). This gives rise to a system of two coupled second order equations in ρ-derivatives, whose principal part is, in matricial form,

$$\frac{(l-1)(l+2)}{2r^2}\;\frac{1}{\alpha}\begin{pmatrix} \frac{2l(l+1)}{a}Q_1 \\ P_1 \end{pmatrix}_{,t} = \begin{pmatrix} -\Pi_1 & -1 \\ \Pi_1^2 & \Pi_1 \end{pmatrix}\;\frac{1}{a^2}\begin{pmatrix} \frac{2l(l+1)}{a}Q_1 \\ P_1 \end{pmatrix}_{,\rho\rho} + \dots \tag{65}$$

the dots denoting lower order terms in ρ-derivatives of Q_1 and P_1. We have divided Q_1 by a to make it a scalar under changes of ρ coordinate. This is a second order in time evolution system, as corresponds to a single wave-like degree of freedom, but it apparently has fourth order in ρ-derivatives for generic values of the background variable Π_1. This is false because the 2x2 matrix has always vanishing square, and hence the system has third order at most. Actually it has second order, as can be checked by taking the matrix to its Jordan canonical form. Defining the combination

$$L \equiv P_1 + l(l+1)\frac{2\Pi_1}{a}Q_1 \tag{66}$$

the system (65) is equivalent to the pair (we now use the dot and prime frame derivatives to simplify the expressions):

$$(r^2 L)^{\cdot} = -2\lambda\frac{Q_1}{a}, \tag{67}$$

$$\left(-\frac{2\lambda}{r^2}\frac{Q_1}{a}\right)^{\cdot} = \frac{1}{\alpha}\left(\frac{\alpha}{r^2}(r^2 L)'\right)' + \frac{\Pi_2 - \Pi_1}{2r^2}(r^2 L)^{\cdot} - \frac{(l-1)(l+2)}{r^2}L, \tag{68}$$

which can be clearly combined into a single second order equation for L, the sought generalization of the Regge-Wheeler equation for dynamical backgrounds.

When restricting to vacuum the variable rL/λ is the Cunningham-Price-Moncrief master function [7] that obeys the Regge-Wheeler equation, though it is not immediately related to the Regge-Wheeler variable. Using the gauge $r = \rho, \beta = 0$ in vacuum we have $\Pi_1 = 0$ and hence $rL = rP_1$, while the Regge-Wheeler variable is $Q_1/(a^2 r)$. We have seen that the former is easily generalizable to a dynamical situation as given in (66), but not the latter, because it would require dividing by Π_1, which may vanish.

D. The master scalar perturbation

The physical variables Q_1 and P_1 are not scalars in M^2 and therefore their values depend upon the foliation we have chosen. It is better to describe the gravitational wave using not only gauge-invariant information, but also foliation-invariant information, that is, scalars in M^2. Studying the geometric properties of those variables under changes of foliation is not simple in the 3+1 notation. Following Gerlach and Sengupta [10], we change to the M^2-adapted framework introduced in Section III, in which the foliation is described by the orthonormal frame (u^A, n^B). This allows us to look for scalars on M^2 as expressions which are frame independent. The background momenta can be rewritten as

$$\Pi_1 = -2u^A v_A, \tag{69}$$

$$\Pi_2 = -2u^A v_A - 2u^A{}_{|A}, \tag{70}$$

$$\Pi_3 = u^A \Phi_{,A}. \tag{71}$$

We see that Π_1 and Π_3 are essentially time components of vectors in M^2. However Π_2 is a more complicated object.

In the GS formalism the axial part of the metric perturbation is decomposed in tensor spherical harmonics:

$$\delta\left(g_{\mu\nu}\right) dx^\mu dx^\nu \equiv h_{\mu\nu} dx^\mu dx^\nu \equiv \sum_{l,m} \left\{ (h_A)_l^m \; X_l^m{}_b dx^A dx^b + h_l^m \; X_l^m{}_{ab} dx^a dx^b \right\}. \tag{72}$$

For a given pair (l, m) the vector h_A and scalar h are related in the following way to the original Hamiltonian variables (35),

$$\begin{aligned} h_1 &= -(h_\rho)_{GS}, & (73)\\ h_2 &= 2(h)_{GS}, & (74)\\ h_0 &= \alpha^2 (h^t)_{GS}. & (75) \end{aligned}$$

The perturbations h_A and h are gauge-dependent, but the following combination is gauge-invariant

$$\kappa_A \equiv h_A - r^2 \left(\frac{h}{r^2}\right)_{,A}, \tag{76}$$

and fully contains the axial information. Therefore it must be given in terms of the gauge-invariant variables Q_1, P_1 and P_2:

$$\begin{aligned} \kappa_\rho &= -Q_1, & (77)\\ \kappa^t &= \frac{1}{\alpha}\left[\hat{p}_2 + \frac{\Pi_2}{2} h_2\right] = \frac{1}{\lambda a \alpha}\left\{ r^2 P_2 - \frac{1}{2}\left[r^2 P_1 + 2l(l+1)\frac{r^2 \Pi_1}{a} Q_1 \right]_{,\rho} \right\}. & (78) \end{aligned}$$

Those relations can be inverted, giving

$$\begin{aligned} Q_1 &= -\kappa_\rho, & (79)\\ Q_2 &= 2(h)_{GS}, & (80)\\ \frac{P_1}{l(l+1)} &= -\epsilon^{AB}\kappa_{A,B} - 2(n^A u^B + n^B u^A) v_A \kappa_B, & (81)\\ \frac{2}{a\alpha}\frac{r^2 P_2}{l(l+1)} &= (l-1)(l+2)\kappa^t + \epsilon^{tC}\left\{ r^4 \epsilon^{AB}\left[r^{-2}\kappa_A\right]_{,B} \right\}_{,C}. & (82) \end{aligned}$$

We see that the gauge-invariant Q_1 is the ρ component of the gauge-invariant vector $-\kappa_A$. Then Q_2 is a scalar in M^2, but it is gauge-dependent. The momentum P_1 is the sum of two parts, the first one being a scalar (the curl of the vector κ_A) and the second one being essentially the off-diagonal component of the symmetric tensor $v_{(A}\kappa_{B)}$. Therefore P_1 is not a component of a tensor itself. Finally P_2 is, apart from a factor $a\alpha$, the time component of a contravariant vector. Again, it is important to stress that these properties are very easy to obtain in GS formalism, but not in the original Hamiltonian formalism, where the variables are well adapted to a 3d point of view.

We want to construct a scalar from a linear combination of Q_1 and P_1, and perhaps their radial derivatives. We already know that P_1 is the sum of a scalar plus a non-scalar, so that we have to find whether it is possible to cancel that non-scalar part using Q_1. It is possible because:

$$-2(n^A u^B + n^B u^A) v_A \kappa_B = -2\epsilon^{AB} v_A \kappa_B - 4(u^A v_A)(n^B \kappa_B). \tag{83}$$

The first term on the r.h.s. is a scalar and the second term is just $-2\Pi_1 \frac{Q_1}{a}$.

Therefore the linear combination L (66) we defined in the previous section is the scalar we were looking for. It is related to the GS scalar variable as

$$-\frac{1}{r^2}\frac{L}{l(l+1)} = \Pi_{GS} \equiv \epsilon^{AB}\left[r^{-2}\kappa_A\right]_{,B}. \tag{84}$$

This is the most important result of this paper: we have constructed a gauge-invariant variable fully describing the physical content of the axial gravitational wave and then we have shown that it is a scalar, so that it is also independent of the coordinate system used on the background spacetime. Note that no other independent scalar can be formed as a linear combination of P_1 and Q_1 and their derivatives. For the cases when we Π_1 vanishes, which is equivalent to the background gauge condition $\rho = r$ and $\beta = 0$ [see Eq. (30)], the variable P_1 is already a scalar. Hence, it is again clear that P_1 is a more convenient variable than Q_1 for these cases.

Gerlach and Sengupta showed [10] that the master variable (84) obeys the following wave equation

$$-\left[\frac{1}{2r^2}(r^4\Pi_{GS})^{|A}\right]_{|A} + \frac{(l-1)(l+2)}{2}\Pi_{GS} = 8\pi\epsilon^{AB}\psi_{A|B}, \tag{85}$$

where the bar denotes the covariant derivative on the manifold M^2 and ψ_A is a gauge-invariant axial perturbation of the energy-momentum tensor. This equation, equivalent to the pair (67–68) for scalar field matter, is valid for all background spherical spacetimes and describes the evolution of the axial gravitational wave coupled to any matter model. Of course, different matter models will have additional variables and equations coupled to (85) but we stress the fact that both the form of Π_{GS} and the form of (85) will remain unchanged.

The only issue we still need to analyze is whether the vector field κ_A, which could appear in the ψ_A expression or in those other equations for the matter variables, can always be reconstructed from the Π_{GS} scalar. An important equation in the GS formalism is

$$(l-1)(l+2)\kappa_A = 16\pi r^2\psi_A - \epsilon_{AB}(r^4\Pi_{GS})^{|B}, \tag{86}$$

whose time component gives (82), after imposing the constraint $P_2 = 0$, and its radial component gives (67) for scalar field matter. This equation can be solved algebraically for κ_A in terms of Π_{GS} as long as ψ_A does not contain the symmetrized derivative $\kappa_{(A|B)}$ or higher derivatives of κ_A. Second and higher derivatives of κ_A can be ruled out by requiring that the matter stress-energy momentum must not contain second derivatives of the metric, because that would change the principal part of the Einstein equations. Symmetrized first order derivatives of κ_A cannot be ruled out on physical grounds because perturbation of covariant derivatives of tensor fields may introduce the term

$$\delta^{axial}[\Gamma^a{}_{BC}] = X^a\,\frac{\kappa_{(B|C)}}{r^2}. \tag{87}$$

This is the only possible source of symmetrized derivatives of κ_A; all other perturbations of Christoffel give either Π_{GS} or undifferentiated κ_A terms. However, we haven't found any standard matter model in which (87) appears under perturbation of its stress-energy tensor.

The combination of Hamiltonian gauge methods with the imposition of having a scalar field on M^2 has determined uniquely the Gerlach and Sengupta scalar Π_{GS}, a variable containing all information on the axial gravitational wave and obeying a master wave equation. There are a number of ways of explaining the meaning of this variable (see for instance [17]), but probably the simplest one is given by the expression

$$\delta\left(R_{ABcd}\right) = -\frac{l(l+1)}{2}\,Y\,\epsilon_{AB}\,\epsilon_{cd}\,r^2\Pi_{GS}, \tag{88}$$

where Y is the scalar harmonic.

V. CONCLUSIONS

The use of linear perturbation theory avoids the intrinsic nonlinear character of General Relativity, but introduces new problems that must always be addressed in some form. A Hamiltonian approach to GR perturbations allows a natural discrimination of the gauge from the gauge-invariant information in the problem, but obscures the geometric character of the perturbative variables. A complementary Lagrangian approach offers a better geometric understanding, but gives no clues on how to separate the physical, unconstrained content of the model at hand. This Article proposes a combination of those two approaches to arrive at a scalar, gauge-invariant and unconstrained description of the linear perturbations in General Relativity.

Around a spherically symmetric background the axial and polar subsets of perturbations decouple from each other; this Article has focused on the axial subset, for which a metric master scalar was already introduced by Gerlach and Sengupta [10] using a purely Lagrangian approach. Generalizing Moncrief's approach for a Schwarzschild background [4], we have reobtained this scalar for a general time-dependent background. First, we have isolated the gauge-invariant and unconstrained information in a Hamiltonian pair of variables (Q_1, P_1), each obeying a first order in time evolution equation. Then we have analyzed the geometric character of these two variables, showing that only a particular combination of them forms a scalar under transformations on the reduced (under spherical symmetry) spacetime. This scalar is, as was expected, the Gerlach and Sengupta master scalar.

The corresponding master scalar for the polar sector has never been found for a general time-dependent background, and there is no known obstruction for its existence. Such a variable would be relevant to study, for example, the matching problem through a timelike surface separating two different physical models (like fluid and vacuum at a star

surface [18]). Therefore, the open question is whether the same combination of techniques we have used in this article can be successfully applied to the polar gravitational wave. This analysis will be presented in a separate publication.

Many computations in this Article have been performed with the *xAct* [19, 20, 21] framework for Tensor Computer Algebra. More precisely, we have used the package *xPert* for metric perturbation theory around general spacetimes to obtain the formulas of Section II. Then, we have expanded these formulas in tensor spherical harmonics with the package *Harmonics*.

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APPENDIX A: TENSOR SPHERICAL HARMONICS

It is possible to construct harmonic bases for tensor fields of arbitrary rank on the two-sphere S^2. This appendix briefly summarizes the construction of those bases for scalars, vectors and symmetric tensors. For the general case see [1]. Coordinates on S^2 will be denoted with lowercase Latin letters $a, b, c, ...$ and the round metric will be γ_{ab}, with unit Gaussian curvature. Its associated Levi-Civita connection will be denoted with a colon, such that $\gamma_{ab:c} = 0$. Finally, the volume form will be called ϵ_{ab}.

The spherical harmonics Y_l^m form a basis for scalar fields on S^2. A vector basis can be constructed from them by differentiation, following Regge and Wheeler, as $Z_a \equiv Y_{:a}$ and $X_a \equiv \epsilon_{ab}\gamma^{bc}Y_{:c}$, the former being polar and the latter axial. Note that the labels (l, m) are always implicitly assumed in the equations. A basis for polar symmetric tensors is given by $\gamma_{ab}Y$ (pure trace) and $Z_{ab} \equiv Y_{:ab} + \frac{l(l+1)}{2}\gamma_{ab}Y$ (traceless). Axial symmetric tensors can be expanded using the basis $X_{ab} \equiv (X_{a:b} + X_{b:a})/2$.

These tensor harmonics are related to those of Moncrief [4] as follows:

$$(\hat{e}_1)_{ij}dx^i dx^j = -2d\rho\, X_a dx^a, \tag{A1}$$
$$(\hat{e}_2)_{ij}dx^i dx^j = X_{ab}dx^a dx^b, \tag{A2}$$
$$(\hat{f}_1)_{ij}dx^i dx^j = 2d\rho\, Z_a dx^a, \tag{A3}$$
$$(\hat{f}_2)_{ij}dx^i dx^j = d\rho^2\, Y, \tag{A4}$$
$$(\hat{f}_3)_{ij}dx^i dx^j = Y\gamma_{ab}\, dx^a dx^b, \tag{A5}$$
$$(\hat{f}_4)_{ij}dx^i dx^j = Y_{a:b}\, dx^a dx^b. \tag{A6}$$

Under integration over the two-sphere the tensor spherical harmonics form an orthogonal basis, with the following normalizations:

$$\int_{S^2} d\Omega\, Y^*\, Y' = \delta_{ll'}\delta_{mm'}, \tag{A7}$$
$$\int_{S^2} d\Omega\, \gamma^{ab}\, Z_a^*\, Z_b' = l(l+1)\delta_{ll'}\delta_{mm'}, \tag{A8}$$
$$\int_{S^2} d\Omega\, \gamma^{ab}\, X_a^*\, X_b' = l(l+1)\delta_{ll'}\delta_{mm'}, \tag{A9}$$
$$\int_{S^2} d\Omega\, \gamma^{ab}\gamma^{cd}\, Z_{ac}^*\, Z_{bd}' = \frac{1}{2}\frac{(l+2)!}{(l-2)!}\delta_{ll'}\delta_{mm'}, \tag{A10}$$
$$\int_{S^2} d\Omega\, \gamma^{ab}\gamma^{cd}\, X_{ac}^*\, X_{bd}' = \frac{1}{2}\frac{(l+2)!}{(l-2)!}\delta_{ll'}\delta_{mm'}. \tag{A11}$$

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